

# An example of algebraization of analysis and Fibonacci cobweb poset characterization

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## Abstract

In [10, 17] inspired by O. V. Viskov [23] it was shown that the  $\psi$ -calculus in parts appears to be almost automatic, natural extension of classical operator calculus of Rota - Mullin or equivalently - of umbral calculus of Roman and Rota. At the same time this calculus is an example of the algebraization of the analysis - here restricted to the algebra of polynomials. The first part of the article is the review of the recent author's contribution [4]. The main definitions and theorems of Finite Fibonomial Operator Calculus which is a special case of  $\psi$ -extended Rota's finite operator calculus [9, 10] are presented there. In the second part the characterization of Fibonacci Cobweb poset  $P$  as DAG and oDAG is given. The  $\dim 2$  poset such that its Hasse diagram coincide with digraf of  $P$  is constructed.

KEY WORDS: Extended umbral calculus, Fibonomial calculus, Fibonacci cobweb poset, DAG

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## 1 Introduction

In [14] it was shown that: „any  $\psi$ -representation of finite operator calculus or equivalently - any  $\psi$ -representation of GHW algebra makes up an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials.(...) Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on  $P$  ( the algebra of polynomials) are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited. For example  $\frac{d}{dx} = \sum_{k \geq 1} \frac{d_k}{k!} \Delta^k$  where  $d_k = \left[ \frac{d}{dx} x^k \right]_{x=0} = (-1)^{k-1} (k-1)!$  or  $\Delta = \sum_{n \geq 1} \frac{\delta_n}{n!} \frac{d^n}{dx^n}$  where  $\delta_n = [\Delta x^n]_{x=0} = 1$ . Thus the difference and differential operators and equations are treated on

the same footing.” The authors goal was there „to deliver the general scheme of ” $\psi$ -umbral” algebraization of the analysis of general differential operators [19].” One may consider a plenty of different special cases of these  $\psi$ -extensions. Each of them can be obtained by the special choice of an admissible sequence  $\psi = \{\psi_n(q)\}_{n \geq 0}$  ;  $\psi_n(q) \neq 0$ ;  $n \geq 0$  and  $\psi_0(q) = 1$ , or equivalently, by the special choice of an sequence  $n_\psi$ , where ([8, 9, 10, 14])

$$n_\psi \equiv \psi_{n-1}(q) \psi_n^{-1}(q), n \geq 0,$$

and each of them constitutes a representation of GHW algebra and provides an example of the algebraization of the analysis.

One of most interesting cases is the so called Finite Fibonomial Operator Calculus (FFOC). Its idea comes from [9] and it was considered by the present author in [4]. It is also the main object of this work. In the first part of it, we present some definitions and theorems of FFOC. We take there  $n_\psi = n_F = F_n$ , where the famous Fibonacci sequence  $\{F_n\}_{n \geq 0}$

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_0 = 0, F_1 = 1 \end{cases}$$

is attributed and referred to the first edition (lost) of ”Liber Abaci” (1202) by Leonardo Fibonacci (Pisano)(see edition from 1228 reproduced as ”Il Liber Abaci di Leonardo Pisano pubblicato secondo la lezione Codice Magliabeciano by Baldassarre Boncompagni in Scritti di Leonardo Pisano” , vol. 1, (1857) Rome).

In order to formulate main results of FFOC the following objects for the sequence  $F = \{F_n\}_{n \geq 0}$  are defined:

(1)  $F$ -factorial:

$$F_n! = F_n F_{n-1} \dots F_2 F_1, \quad F_0! = 1.$$

(2)  $F$ -binomial (Fibonomial ) coefficients [2]:

$$\binom{n}{k}_F = \frac{n_F!}{k_F!} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_k F_{k-1} \dots F_2 F_1} = \frac{F_n!}{F_k! F_{n-k}!}, \quad \binom{n}{0}_F = 1.$$

It is known that  $\binom{n}{k}_F \in \mathbf{N}$  for every  $n, k \in \mathbf{N} \cup 0$ .

In [16] some applications of  $\psi$ -extensions of the umbral calculus (including FFOC) were presented. As annouced there, the combinatorial interpretation of fibonomial coefficients has been found by A. K. Kwaśniewski in [15]. It was done by the use of the so called Fibonacci cobweb poset [11, 12, 13, 15, 17, 18]. In [5, 6] the incidence algebra of the Fibonacci cobweb poset was considered by the present author. In the second part of this work the characterisation of this poset  $P$  as DAG and oDAG is given. The  $\dim 2$  poset such that its Hasse diagram coincide with digraf of  $P$  is constructed. Directed acyclic graphs (DAGs) have many important applications in computer science, including: the parse

tree constructed by a compiler, a reference graph that can be garbage collected using simple reference counting, dependency graphs such as those used in instruction scheduling and makefiles, dependency graphs between classes formed by inheritance relationships in object-oriented programming languages. In theoretical physics a directed acyclic graph can be used to represent spacetime as a causal set. In bioinformatics, DAGs can be used to find areas of syntenicity between two genomes. They can be also used in abstract process descriptions such as workflows and some models of provenance.

## 2 Finite Fibonomial Operator Calculus

### 2.1 Operators and polynomial sequences

Let  $\mathbf{P}$  be the algebra of polynomials over the field  $\mathbb{K}$  of characteristic zero.

**Definition 2.1.** *The linear operator  $\partial_F : \mathbf{P} \rightarrow \mathbf{P}$  such that  $\partial_F x^n = F_n x^{n-1}$  for  $n \geq 0$  is named the  $F$ -derivative.*

**Definition 2.2.** *The  $F$ -translation operator is the linear operator  $E^y(\partial_F) : \mathbf{P} \rightarrow \mathbf{P}$  of the form:*

$$E^y(\partial_F) = \exp_F\{y\partial_F\} = \sum_{k \geq 0} \frac{y^k \partial_F^k}{F_k!}, \quad y \in \mathbf{K}$$

**Definition 2.3.**

$$\forall_{p \in \mathbf{P}} \quad p(x +_F y) = E^y(\partial_F)p(x) \quad x, y \in \mathbf{K}$$

**Definition 2.4.** *A linear operator  $T : \mathbf{P} \rightarrow \mathbf{P}$  is said to be  $\partial_F$ -shift invariant iff*

$$\forall_{y \in \mathbf{K}} \quad [T, E^y(\partial_F)] = TE^y(\partial_F) - E^y(\partial_F)T = 0$$

*We shall denote by  $\Sigma_F$  the algebra of  $F$ -linear  $\partial_F$ -shift invariant operators.*

**Definition 2.5.** *Let  $Q(\partial_F)$  be a formal series in powers of  $\partial_F$  and  $Q(\partial_F) : \mathbf{P} \rightarrow \mathbf{P}$ .  $Q(\partial_F)$  is said to be  $\partial_F$ -delta operator iff*

- (a)  $Q(\partial_F) \in \Sigma_F$
- (b)  $Q(\partial_F)(x) = \text{const} \neq 0$

Under quite natural specification the proofs of most statements might be referred to [8](see also references therein).

The particularities of the case considered here are revealed in the sequel. There the scope of new possibilities is initiated by means of unknown before examples.

**Proposition 2.1.** *Let  $Q(\partial_F)$  be the  $\partial_F$ -delta operator. Then*

$$\forall_{c \in \mathbf{K}} \quad Q(\partial_F)c = 0.$$

**Proposition 2.2.** Every  $\partial_F$ -delta operator reduces degree of any polynomial by one.

**Definition 2.6.** The polynomial sequence  $\{q_n(x)\}_{n \geq 0}$  such that  $\deg q_n(x) = n$  and:

- (1)  $q_0(x) = 1$ ;
- (2)  $q_n(0) = 0, n \geq 1$ ;
- (3)  $Q(\partial_F)q_n(x) = F_n q_{n-1}(x), n \geq 0$

is called  $\partial_F$ -basic polynomial sequence of the  $\partial_F$ -delta operator  $Q(\partial_F)$ .

**Proposition 2.3.** For every  $\partial_F$ -delta operator  $Q(\partial_F)$  there exists the uniquely determined  $\partial_F$ -basic polynomial sequence  $\{q_n(x)\}_{n \geq 0}$ .

**Definition 2.7.** A polynomial sequence  $\{p_n(x)\}_{n \geq 0}$  ( $\deg p_n(x) = n$ ) is of  $F$ -binomial (fibonomial) type if it satisfies the condition

$$E^y(\partial_F)p_n(x) = p_n(x +_F y) = \sum_{k \geq 0} \binom{n}{k}_F p_k(x) p_{n-k}(y) \quad \forall y \in \mathbf{K}$$

**Theorem 2.1.** The polynomial sequence  $\{p_n(x)\}_{n \geq 0}$  is a  $\partial_F$ -basic polynomial sequence of some  $\partial_F$ -delta operator  $Q(\partial_F)$  iff it is a sequence of  $F$ -binomial type.

**Theorem 2.2.** (First Expansion Theorem)

Let  $T \in \Sigma_F$  and let  $Q(\partial_F)$  be a  $\partial_F$ -delta operator with  $\partial_F$ -basic polynomial sequence  $\{q_n\}_{n \geq 0}$ . Then

$$T = \sum_{n \geq 0} \frac{a_n}{F_n!} Q(\partial_F)^n; \quad a_n = [Tq_n(x)]_{x=0}.$$

**Theorem 2.3.** (Isomorphism Theorem)

Let  $\Phi_F = \mathbf{K}_F[[t]]$  be the algebra of formal  $\exp_F$  series in  $t \in \mathbf{K}$ , i.e.:

$$f_F(t) \in \Phi_F \quad \text{iff} \quad f_F(t) = \sum_{k \geq 0} \frac{a_k t^k}{F_k!} \quad \text{for} \quad a_k \in \mathbf{K},$$

and let the  $Q(\partial_F)$  be a  $\partial_F$ -delta operator. Then  $\Sigma_F \approx \Phi_F$ . The isomorphism  $\phi: \Phi_F \rightarrow \Sigma_F$  is given by the natural correspondence:

$$f_F(t) = \sum_{k \geq 0} \frac{a_k t^k}{F_k!} \xrightarrow{\text{into}} T_{\partial_F} = \sum_{k \geq 0} \frac{a_k}{F_k!} Q(\partial_F)^k.$$

**Remark 2.1.** In the algebra  $\Phi_F$  the product is given by the fibonomial convolution, i.e.:

$$\left( \sum_{k \geq 0} \frac{a_k}{F_k!} x^k \right) \left( \sum_{k \geq 0} \frac{b_k}{F_k!} x^k \right) = \left( \sum_{k \geq 0} \frac{c_k}{F_k!} x^k \right)$$

where

$$c_k = \sum_{l \geq 0} \binom{k}{l}_F a_l b_{k-l}.$$

**Corollary 2.1.** *Operator  $T \in \Sigma_F$  has its inverse  $T^{-1} \in \Sigma_\psi$  iff  $T1 \neq 0$ .*

**Remark 2.2.** The  $F$ -translation operator  $E^y(\partial_F) = \exp_F\{y\partial_F\}$  is invertible in  $\Sigma_F$  but it is not a  $\partial_F$ -delta operator. No one of  $\partial_F$ -delta operators  $Q(\partial_F)$  is invertible with respect to the formal series "F-product".

**Corollary 2.2.** *Operator  $R(\partial_F) \in \Sigma_F$  is a  $\partial_F$ -delta operator iff  $a_0 = 0$  and  $a_1 \neq 0$ , where  $R(\partial_F) = \sum_{n \geq 0} \frac{a_n}{F_n!} Q(\partial_F)^n$  or equivalently :  $r(0) = 0$  &  $r'(0) \neq 0$  where  $r(x) = \sum_{k \geq 0} \frac{a_k}{F_k!} x^k$  is the correspondent of  $R(\partial_F)$  under the Isomorphism Theorem.*

**Corollary 2.3.** *Every  $\partial_F$ -delta operator  $Q(\partial_F)$  is a function  $Q(\partial_F)$  according to the expansion*

$$Q(\partial_F) = \sum_{n \geq 1} \frac{q_n}{F_n!} \partial_F^n$$

*This  $F$ -series will be called the  $F$ -indicator of the  $Q(\partial_F)$ .*

**Remark 2.3.**  $\exp_F\{zx\}$  is the  $F$ -exponential generating function for  $\partial_F$ -basic polynomial sequence  $\{x^n\}_{n=0}^\infty$  of the  $\partial_F$  operator.

**Corollary 2.4.** *The  $F$ -exponential generating function for  $\partial_F$ -basic polynomial sequence  $\{p_n(x)\}_{n=0}^\infty$  of the  $\partial_F$ -delta operator  $Q(\partial_F)$  is given by the following formula*

$$\sum_{k \geq 0} \frac{p_k(x)}{F_k!} z^k = \exp_F\{xQ^{-1}(z)\}$$

where

$$Q \circ Q^{-1} = Q^{-1} \circ Q = I = id.$$

**Example 2.1.** The following operators are the examples of  $\partial_F$ -delta operators:

- (1)  $\partial_F$ ;
- (2)  $F$ -difference operator  $\Delta_F = E^1(\partial_F) - I$  such that  $(\Delta_F p)(x) = p(x +_F 1) - p(x)$  for every  $p \in \mathbf{P}$  ;
- (3) The operator  $\nabla_F = I - E^{-1}(\partial_F)$  defined as follows:  $(\nabla_F p)(x) = p(x) - p(x -_F 1)$  for every  $p \in \mathbf{P}$ ;
- (4)  $F$ -Abel operator:  $A(\partial_F) = \partial_F E^a(\partial_F) = \sum_{k \geq 0} \frac{a^k}{F_k!} \partial_F^{k+1}$ ;
- (5)  $F$ -Laguerre operator of the form:  $L(\partial_F) = \frac{\partial_F}{\partial_F - I} = \sum_{k \geq 0} \partial_F^{k+1}$ .

**Definition 2.8.** The  $\hat{x}_F$ -operator is the linear map  $\hat{x}_F : \mathbf{P} \rightarrow \mathbf{P}$  such that  $\hat{x}_F x^n = \frac{n+1}{F_{n+1}} x^{n+1}$  for  $n \geq 0$ . ( $[\partial_F, \hat{x}_F] = id$ .)

**Definition 2.9.** A linear map  $' : \Sigma_F \rightarrow \Sigma_F$  such that  $T' = T \hat{x}_F - \hat{x}_F T = [T, \hat{x}_F]$  is called the Graves-Pincherle  $F$ -derivative  $[?, ?]$ .

**Example 2.2.**

$$(1) \partial_F' = I = id;$$

$$(2) (\partial_F)^{n'} = n \partial_F^{n-1}$$

According to the example above the Graves-Pincherle  $F$ -derivative is the formal derivative with respect to  $\partial_F$  in  $\Sigma_F$  i.e.,  $T' (\partial_F) \in \Sigma_F$  for any  $T \in \Sigma_F$ .

**Corollary 2.5.** Let  $t(z)$  be the indicator of operator  $T \in \Sigma_F$ . Then  $t'(z)$  is the indicator of  $T' \in \Sigma_F$ .

Due to the isomorphism theorem and the Corollaries above the Leibnitz rule holds .

**Proposition 2.4.**  $(TS)' = T' S + ST'$  ;  $T, S \in \Sigma_F$ .

As an immediate consequence of the Proposition 2.4 we get

$$(S^n)' = n S' S^{n-1} \quad \forall S \in \Sigma_F.$$

From the isomorphism theorem we insert that the following is true.

**Proposition 2.5.**  $Q(\partial_F)$  is the  $\partial_F$ -delta operator iff there exists invertible  $S \in \Sigma_F$  such that

$$Q(\partial_F) = \partial_F S.$$

The Graves-Pincherle  $F$ -derivative notion appears very effective while formulating expressions for  $\partial_F$ -basic polynomial sequences of the given  $\partial_F$ -delta operator  $Q(\partial_F)$ .

**Theorem 2.4.** ( $F$ -Lagrange and  $F$ -Rodrigues formulas) [8, 22, 19]  
Let  $\{q_n\}_{n \geq 0}$  be  $\partial_F$ -basic sequence of the delta operator  $Q(\partial_F)$ ,  $Q(\partial_F) = \partial_F P$  ( $P \in \Sigma_F$ , invertible). Then for  $n \geq 0$ :

$$(1) q_n(x) = Q(\partial_F)' P^{-n-1} x^n ;$$

$$(2) q_n(x) = P^{-n} x^n - \frac{F_n}{n} (P^{-n})' x^{n-1};$$

$$(3) q_n(x) = \frac{F_n}{n} \hat{x}_F P^{-n} x^{n-1};$$

$$(4) q_n(x) = \frac{F_n}{n} \hat{x}_F (Q(\partial_F)')^{-1} q_{n-1}(x) \quad (\leftarrow \text{Rodrigues } F\text{-formula}).$$

**Corollary 2.6.** Let  $Q(\partial_F) = \partial_F S$  and  $R(\partial_F) = \partial_F P$  be the  $\partial_F$ -delta operators with the  $\partial_F$ -basic sequences  $\{q_n(x)\}_{n \geq 0}$  and  $\{r_n(x)\}_{n \geq 0}$  respectively. Then:

- (1)  $q_n(x) = R'(Q')^{-1}S^{-n-1}P^{n+1}r_n(x), \quad n \geq 0;$
- (2)  $q_n(x) = \hat{x}_F(PS^{-1})^n\hat{x}_F^{-1}r_n(x), \quad n > 0.$

The formulas of the Theorem 2.4 can be used to find  $\partial_F$ -basic sequences of the  $\partial_F$ -delta operators from the Example 2.1.

**Example 2.3.**

- (1) The polynomials  $x^n, \quad n \geq 0$  are  $\partial_F$ -basic for  $F$ -derivative  $\partial_F$ .
- (2) Using Rodrigues formula in a straightforward way one can find the following first  $\partial_F$ -basic polynomials of the operator  $\Delta_F$ :
 
$$\begin{aligned} q_0(x) &= 1 \\ q_1(x) &= x \\ q_2(x) &= x^2 - x \\ q_3(x) &= x^3 - 4x^2 + 3x \\ q_4(x) &= x^4 - 9x^3 + 24x^2 - 16x \\ q_5(x) &= x^5 - 20x^4 + 112.5x^3 - 250x^2 + 156.5x \\ q_6(x) &= x^6 - 40x^5 + 480x^4 - 2160x^3 + 4324x^2 - 2605x. \end{aligned}$$
- (3) Analogously to the above example we find the following first  $\partial_F$ -basic polynomials of the operator  $\nabla_F$ :
 
$$\begin{aligned} q_0(x) &= 1 \\ q_1(x) &= x \\ q_2(x) &= x^2 + x \\ q_3(x) &= x^3 + 4x^2 + 3x \\ q_4(x) &= x^4 + 9x^3 + 24x^2 + 16x \\ q_5(x) &= x^5 + 20x^4 + 112.5x^3 + 250x^2 + 156.5x \\ q_6(x) &= x^6 + 40x^5 + 480x^4 + 2160x^3 + 4324x^2 + 2605x. \end{aligned}$$
- (4) Using Rodrigues formula in a straightforward way one finds the following first  $\partial_F$ -basic polynomials of  $F$ -Abel operator:
 
$$\begin{aligned} A_{0,F}^{(a)}(x) &= 1 \\ A_{1,F}^{(a)}(x) &= x \\ A_{2,F}^{(a)}(x) &= x^2 + ax \\ A_{3,F}^{(a)}(x) &= x^3 - 4ax^2 + 2a^2x \\ A_{4,F}^{(a)}(x) &= x^4 - 9ax^3 + 18a^2x^2 - 3a^3x. \end{aligned}$$
- (5) In order to find  $\partial_F$ -basic polynomials of  $F$ -Laguerre operator  $L(\partial_F)$  we

use formula (3) from Theorem 2.4:

$$\begin{aligned}
L_{n,F}(x) &= \frac{F_n}{n} \hat{x}_F \left( \frac{1}{\partial_F - 1} \right)^{-n} x^{n-1} = \frac{F_n}{n} \hat{x}_F (\partial_F - 1)^n x^{n-1} = \\
&= \frac{F_n}{n} \hat{x}_F \sum_{k=0}^n (-1)^k \binom{n}{k} \partial_F^{n-k} x^{n-1} = \frac{F_n}{n} \hat{x}_F \sum_{k=0}^n (-1)^k \binom{n}{k} (n-1)_F^{n-k} x^{k-1} = \\
&= \frac{F_n}{n} \sum_{k=1}^n (-1)^k \binom{n}{k} (n-1)_F^{n-k} \frac{k}{F_k} x^k.
\end{aligned}$$

## 2.2 Sheffer $F$ -polynomials

**Definition 2.10.** A polynomial sequence  $\{s_n\}_{n \geq 0}$  is called the sequence of Sheffer  $F$ -polynomials of the  $\partial_F$ -delta operator  $Q(\partial_F)$  iff

- (1)  $s_0(x) = \text{const} \neq 0$
- (2)  $Q(\partial_F)s_n(x) = F_n s_{n-1}(x); n \geq 0$ .

**Proposition 2.6.** Let  $Q(\partial_F)$  be  $\partial_F$ -delta operator with  $\partial_F$ -basic polynomial sequence  $\{q_n\}_{n \geq 0}$ . Then  $\{s_n\}_{n \geq 0}$  is the sequence of Sheffer  $F$ -polynomials of  $Q(\partial_F)$  iff there exists an invertible  $S \in \Sigma_F$  such that  $s_n(x) = S^{-1}q_n(x)$  for  $n \geq 0$ . We shall refer to a given labeled by  $\partial_F$ -shift invariant invertible operator  $S$  Sheffer  $F$ -polynomial sequence  $\{s_n\}_{n \geq 0}$  as the sequence of Sheffer  $F$ -polynomials of the  $\partial_F$ -delta operator  $Q(\partial_F)$  relative to  $S$ .

**Theorem 2.5.** (Second  $F$ -Expansion Theorem)

Let  $Q(\partial_F)$  be the  $\partial_F$ -delta operator  $Q(\partial_F)$  with the  $\partial_F$ -basic polynomial sequence  $\{q_n(x)\}_{n \geq 0}$ . Let  $S$  be an invertible  $\partial_F$ -shift invariant operator and let  $\{s_n(x)\}_{n \geq 0}$  be its sequence of Sheffer  $F$ -polynomials. Let  $T$  be any  $\partial_F$ -shift invariant operator and let  $p(x)$  be any polynomial. Then the following identity holds :

$$\forall_{y \in K} \wedge \forall_{p \in P} \quad (Tp)(x +_F y) = [E^y(\partial_F)p](x) = T \sum_{k \geq 0} \frac{s_k(y)}{F_k!} Q(\partial_F)^k S T p(x).$$

**Corollary 2.7.** Let  $s_n(x)_{n \geq 0}$  be a sequence of Sheffer  $F$ -polynomials of a  $\partial_F$ -delta operator  $Q(\partial_F)$  relative to  $S$ . Then:

$$S^{-1} = \sum_{k \geq 0} \frac{s_k(0)}{F_k!} Q(\partial_F)^k.$$

**Theorem 2.6.** (The Sheffer  $F$ -Binomial Theorem)

Let  $Q(\partial_F)$ , invertible  $S \in \Sigma_F$ ,  $q_n(x)_{n \geq 0}$ ,  $s_n(x)_{n \geq 0}$  be as above. Then:

$$E^y(\partial_F)s_n(x) = s_n(x +_F y) = \sum_{k \geq 0} \binom{n}{k}_F s_k(x) q_{n-k}(y).$$



**Corollary 2.8.**

$$s_n(x) = \sum_{k \geq 0} \binom{n}{k}_F s_k(0) q_{n-k}(x)$$

**Proposition 2.7.** *Let  $Q(\partial_F)$  be a  $\partial_F$ -delta operator. Let  $S$  be an invertible  $\partial_F$ -shift invariant operator. Let  $\{s_n(x)\}_{n \geq 0}$  be a polynomial sequence. Let*

$$\forall_{a \in K} \wedge \forall_{p \in P} \quad E^a(\partial_F)p(x) = \sum_{k \geq 0} \frac{s_k(a)}{F_k!} Q(\partial_F)^k S_{\partial_F} p(x) .$$

*Then the polynomial sequence  $\{s_n(x)\}_{n \geq 0}$  is the sequence of Sheffer  $F$ -polynomials of the  $\partial_F$ -delta operator  $Q(\partial_F)$  relative to  $S$ .*

**Proposition 2.8.** *Let  $Q(\partial_F)$  and  $S$  be as above. Let  $q(t)$  and  $s(t)$  be the indicators of  $Q(\partial_F)$  and  $S$  operators. Let  $q^{-1}(t)$  be the inverse  $F$ -exponential formal power series inverse to  $q(t)$ . Then the  $F$ -exponential generating function of Sheffer  $F$ -polynomials sequence  $\{s_n(x)\}_{n \geq 0}$  of  $Q(\partial_F)$  relative to  $S$  is given by*

$$\sum_{k \geq 0} \frac{s_k(x)}{F_k!} z^k = (s(q^{-1}(z)))^{-1} \exp_F\{xq^{-1}(z)\}.$$

**Proposition 2.9.** *A sequence  $\{s_n(x)\}_{n \geq 0}$  is the sequence of Sheffer  $F$ -polynomials of the  $\partial_F$ -delta operator  $Q(\partial_F)$  with the  $\partial_F$ -basic polynomial sequence  $\{q_n(x)\}_{n \geq 0}$  iff*

$$s_n(x +_F y) = \sum_{k \geq 0} \binom{n}{k}_F s_k(x) q_{n-k}(y) .$$

*for all  $y \in K$*

**Example 2.4.** Hermite  $F$ -polynomials are Sheffer  $F$ -polynomials of the  $\partial_F$ -delta operator  $\partial_F$  relative to invertible  $S \in \Sigma_F$  of the form  $S = \exp_F\{\frac{a\partial_F^2}{2}\}$ . One can get them by formula (see Proposition 2.6):

$$H_{n,F}(x) = S^{-1}x^n = \sum_{k \geq 0} \frac{(-a)^k}{2^k F_k!} n_{F,k}^{2k} x^{n-2k}.$$

**Example 2.5.** Let  $S = (1 - \partial_F)^{-\alpha-1}$ . The Sheffer  $F$ -polynomials of  $\partial_F$ -delta operator  $L(\partial_F) = \frac{\partial_F}{\partial_F - 1}$  relative to  $S$  are Laguerre  $F$ -polynomials of order  $\alpha$ . By Proposition 2.6 we have

$$L_{n,F}^{(\alpha)} = (1 - \partial_F)^{\alpha+1} L_{n,F}(x),$$

From the above formula and using Graves-Pincherle  $F$ -derivative we get

$$L_{n,F}^{(\alpha)}(x) = \sum_{k \geq 0} \frac{F_n!}{F_k!} \binom{\alpha + n}{n - k} (-x)^k$$

for  $\alpha \neq -1$ .

**Example 2.6.** Bernoullie's  $F$ -polynomials of order 1 are Sheffer  $F$ -polynomials of

$\partial_F$ -delta operator  $\partial_F$  related to invertible  $S = \left( \frac{\exp_F\{\partial_F\} - I}{\partial_F} \right)^{-1}$ . Using Proposition 2.6 one arrives at

$$\begin{aligned} B_{n,F}(x) &= S^{-1}x^n = \sum_{k \geq 1} \frac{1}{F_k!} \partial_F^{k-1} x^n = \sum_{k \geq 1} \frac{1}{F_k} \binom{n}{k-1}_F x^{n-k+1} = \\ &= \sum_{k \geq 0} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k} \end{aligned}$$

**Theorem 2.7.** (Reccurence relation for Sheffer  $F$ -polynomials)

Let  $Q, S, \{s_n\}_{n \geq 0}$  be as above. Then the following reccurence formula holds:

$$s_{n+1}(x) = \frac{F_{n+1}}{n+1} \left[ \hat{x}_F - \frac{S'}{S} \right] [Q(\partial_F)']^{-1} s_n(x); \quad n \geq 0.$$

**Example 2.7.** The reccurence formula for the Hermite  $F$ -polynomials is:

$$H_{n+1,F}(x) = \hat{x}_F H_{n,F}(x) - \hat{a}_F F_n H_{n-1,F}(x)$$

**Example 2.8.** The reccurence relation for the Laguerre  $F$ -polynomials is:

$$\begin{aligned} L_{n+1,F}^{(\alpha)}(x) &= -\frac{F_{n+1}}{n+1} [\hat{x}_F - (\alpha+1)(1-\partial_F)^{-1}](\partial_F - 1)^2 L_{n,F}^{(\alpha)}(x) \\ &= \frac{F_{n+1}}{n+1} [\hat{x}_F(\partial_F - 1) + \alpha + 1] L_{n,F}^{(\alpha+1)}(x). \end{aligned}$$

## 2.3 Some examples of $F$ -polynomials

(1) Here are the examples of Laguerre  $F$ -polynomials of order  $\alpha = -1$ :

$$L_{0,F}(x) = 1$$

$$L_{1,F}(x) = -x$$

$$L_{2,F}(x) = x^2 - x$$

$$L_{3,F}(x) = -x^3 + 4x^2 - 2x$$

$$L_{4,F}(x) = x^4 - 9x^3 + 18x^2 - 6x$$

$$L_{5,F}(x) = -x^5 + 20x^4 - 90x^3 + 1280x^2 - 30x$$

$$L_{6,F}(x) = x^6 - 40x^5 + 400x^4 - 1200x^3 + 1200x^2 - 240x$$

$$L_{7,F}(x) = -x^7 + 78x^6 - 1560x^5 + 10400x^4 - 23400x^3 + 18720x^2 -$$

$$- 3120x$$

$$L_{8,F}(x) = x^8 - 147x^7 + 5733x^6 - 76440x^5 + 382200x^4 - 687960x^3 + 458640x^2 - 65520x$$

(2) Here are the examples of Laguerre  $F$ -polynomials of order  $\alpha = 1$ :

$$L_{0,F}^{(1)}(x) = 1$$

$$L_{1,F}^{(1)}(x) = -x + 2$$

$$L_{2,F}^{(1)}(x) = x^2 - 3x + 3$$

$$L_{3,F}^{(1)}(x) = -x^3 + 8x^2 - 12x + 8$$

$$L_{4,F}^{(1)}(x) = x^4 - 15x^3 + 60x^2 - 60x + 30$$

$$L_{5,F}^{(1)}(x) = -x^5 + 30x^4 - 225x^3 + 600x^2 - 450x + 240$$

$$L_{6,F}^{(1)}(x) = x^6 - 56x^5 + 840x^4 - 4200x^3 + 8400x^2 - 5040x + 1680$$

(3) Here we give some examples of the Bernoulli's  $F$ -polynomials of order 1:

$$B_{0,F}(x) = 1$$

$$B_{1,F}(x) = x + 1$$

$$B_{2,F}(x) = x^2 + x + \frac{1}{2}$$

$$B_{3,F}(x) = x^3 + 2x^2 + x + \frac{1}{3}$$

$$B_{4,F}(x) = x^4 + 3x^3 + 3x^2 + x + \frac{1}{5}$$

$$B_{5,F}(x) = x^5 + 5x^4 + \frac{15}{2}x^3 + 5x^2 + x + \frac{1}{8}$$

$$B_{6,F}(x) = x^6 + 8x^5 + 20x^4 + 20x^3 + 8x^2 + x + \frac{1}{13}$$

$$B_{7,F}(x) = x^7 + 13x^6 + 52x^5 + \frac{260}{3}x^4 + 52x^3 + 13x^2 + x + \frac{1}{21}$$

$$B_{8,F}(x) = x^8 + 21x^7 + \frac{273}{2}x^6 + 364x^5 + 364x^4 + \frac{273}{2}x^3 + 21x^2 + x + \frac{1}{36}$$

$$B_{9,F}(x) = x^9 + 34x^8 + 357x^7 + 1547x^6 + \frac{12376}{5}x^5 + 1547x^4 + 357x^3 +$$

$$+ 34x^2 + x + \frac{1}{55}$$

### 3 Fibonacci cobweb poset characterization

#### 3.1 Fibonacci cobweb poset

The Fibonacci cobweb poset  $P$  has been invented by A.K.Kwaśniewski in [15, 11, 12] for the purpose of finding combinatorial interpretation of fibonomial coefficients and eventually their recurrence relation.

In [15] A. K. Kwaśniewski defined cobweb poset  $P$  as infinite labeled digraph oriented upwards as follows: Let us label vertices of  $P$  by pairs of coordinates:  $\langle i, j \rangle \in \mathbf{N}_0 \times \mathbf{N}_0$ , where the second coordinate is the number of level in which the element of  $P$  lies (here it is the  $j$ -th level) and the first one is the number of this element in his level (from left to the right), here  $i$ . Following [15] we shall refer to  $\Phi_s$  as to the set of vertices (elements) of the  $s$ -th level, i.e.:

$$\Phi_s = \{\langle j, s \rangle, 1 \leq j \leq F_s\}, \quad s \in \mathbf{N} \cup \{0\},$$

where  $\{F_n\}_{n \geq 0}$  stands for Fibonacci sequence.

Then  $P$  is a labeled graph  $P = (V, E)$  where

$$V = \bigcup_{p \geq 0} \Phi_p, \quad E = \{\langle \langle j, p \rangle, \langle q, p+1 \rangle \rangle\}, \quad 1 \leq j \leq F_p, \quad 1 \leq q \leq F_{p+1}.$$

We can now define the partial order relation on  $P$  as follows: let  $x = \langle s, t \rangle, y = \langle u, v \rangle$  be elements of cobweb poset  $P$ . Then

$$(x \leq_P y) \iff [(t < v) \vee (t = v \wedge s = u)].$$

#### 3.2 DAG $\longrightarrow$ oDAG problem

In [21] A. D. Plotnikov considered the so called "DAG  $\longrightarrow$  oDAG problem". He determined condition when a digraph  $G$  may be presented by the corresponding  $\dim 2$  poset  $R$  and he established the algorithm for finding it.

Before citing Plotnikov's results let us recall (following [21]) some indispensable definitions.

If  $P$  and  $Q$  are partial orders on the same set  $A$ ,  $Q$  is said to be an **extension** of  $P$  if  $a \leq_P b$  implies  $a \leq_Q b$ , for all  $a, b \in A$ . A poset  $L$  is a **chain**, or a **linear order** if we have either  $a \leq_L b$  or  $b \leq_L a$  for any  $a, b \in A$ . If  $Q$  is a linear order then it is a **linear extension** of  $P$ .

The **dimension**  $\dim R$  of  $R$  being a partial order is the least positive integer  $s$  for which there exists a family  $F = (L_1, L_2, \dots, L_s)$  of linear extensions of  $R$  such that  $R = \bigcap_{i=1}^s L_i$ . A family  $F = (L_1, L_2, \dots, L_s)$  of linear orders on  $A$  is called a **realizer** of  $R$  on  $A$  if

$$R = \bigcap_{i=1}^s L_i.$$

We denote by  $D_n$  the set of all acyclic directed  $n$ -vertex graphs without loops and multiple edges. Each digraph  $\vec{G} = (V, \vec{E}) \in D_n$  will be called **DAG**.

A digraph  $\vec{G} \in D_n$  will be called **orderable (oDAG)** if there exists a  $\dim 2$  poset such that its Hasse diagram coincide with the digraph  $\vec{G}$ .

Let  $\vec{G} \in D_n$  be a digraph, which does not contain the arc  $(v_i, v_j)$  if there exists the directed path  $p(v_i, v_j)$  from the vertex  $v_i$  into the vertex  $v_j$  for any  $v_i, v_j \in V$ . Such digraph is called **regular**. Let  $D \subset D_n$  is the set of all regular graphs.

Let there is a some regular digraph  $\vec{G} = (V, E) \in D$ , and let the chain  $\vec{X}$  has three elements  $x_{i_1}, x_{i_2}, x_{i_3} \in X$  such that  $i_1 < i_2 < i_3$ , and, in the digraph  $\vec{G}$ , there are not paths  $p(v_{i_1}, v_{i_2})$ ,  $p(v_{i_2}, v_{i_3})$  and there exists a path  $p(v_{i_1}, v_{i_3})$ . Such representation of graph vertices by elements of the chain  $\vec{X}$  is called the representation in **inadmissible form**. Otherwise, the chain  $\vec{X}$  presets the graph vertices in **admissible form**.

Plotnikov showed that:

**Lemma 3.1.** [21] *A digraph  $\vec{G} \in D_n$  may be represented by a  $\dim 2$  poset if:*

- (1) *there exist two chains  $\vec{X}$  and  $\vec{Y}$ , each of which is a linear extension of  $\vec{G}_t$ ;*
- (2) *the chain  $\vec{Y}$  is a modification of  $\vec{X}$  with inversions, which remove the ordered pairs of  $\vec{X}$  that there do not exist in  $\vec{G}$ .*

Above lemma results in the algorithm for finding  $\dim 2$  representation of a given DAG (i.e. corresponding oDAG) while the following theorem establishes the conditions for constructing it.

**Theorem 3.1.** [21] *A digraph  $\vec{G} = (V, \vec{E}) \in D_n$  can be represented by  $\dim 2$  poset iff it is regular and its vertices can be presented by the chain  $\vec{X}$  in admissible form.*

### 3.3 Fibonacci cobweb poset as DAG and oDAG

In this section we show that Fibonacci cobweb poset is a DAG and it is orderable (oDAG).

Obviously, cobweb poset  $P = (V, E)$  defined above is a DAG (it is directed acyclic graph without loops and multiple edges). One can also verify that it is regular. For two elements  $\langle i, n \rangle, \langle j, m \rangle \in V$  a directed path  $p(\langle i, n \rangle, \langle j, m \rangle) \notin E$  will exist iff  $n < m + 1$  but then  $(\langle i, n \rangle, \langle j, m \rangle) \notin E$  i.e.  $P$  does not contain the edge  $(\langle i, n \rangle, \langle j, m \rangle)$ .

It is also possible to verify that vertices of cobweb poset  $P$  can be presented in admissible form by the chain  $\vec{X}$  being a linear extension of cobweb  $P$  as follows:

$$\vec{X} = \left( \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle, \langle 3, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 5 \rangle, \langle 3, 5 \rangle, \right. \\ \left. \langle 4, 5 \rangle, \langle 5, 5 \rangle, \dots \right),$$

where

$$(\langle s, t \rangle \leq_{\vec{X}} \langle u, v \rangle) \iff [(s \leq u) \wedge (t \leq v)]$$

for  $1 \leq s \leq F_t, 1 \leq u \leq F_v, t, v \in \mathbf{N} \cup \{0\}$ .

Fibonacci cobweb poset  $P$  satisfies the conditions of Theorem 3.1 so it is oDAG. To find the chain  $\vec{Y}$  being a linear extension of cobweb  $P$  one uses Lemma 3.1 and arrives at:

$$\vec{Y} = (\langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle, \langle 3, 4 \rangle, \langle 2, 4 \rangle, \langle 1, 4 \rangle, \langle 5, 5 \rangle, \langle 4, 5 \rangle, \langle 3, 5 \rangle, \\ \langle 2, 5 \rangle, \langle 1, 5 \rangle, \dots),$$

where

$$(\langle s, t \rangle \leq_{\vec{Y}} \langle u, v \rangle) \iff [(t < v) \vee (t = v \wedge s \geq u)]$$

for  $1 \leq s \leq F_t, 1 \leq u \leq F_v, t, v \in \mathbf{N} \cup \{0\}$  and finally

$$(P, \leq_P) = \vec{X} \cap \vec{Y}.$$

**Remark 3.1.** For any sequence  $\{a_n\}$  of natural numbers one can define corresponding cobweb poset as follows [17]:

$$\Phi_s = \{\langle j, s \rangle, 1 \leq j \leq a_s\}, \quad s \in \mathbf{N} \cup \{0\},$$

and  $P = (V, E)$  where

$$V = \bigcup_{p \geq 0} \Phi_p, \quad E = \{\langle \langle j, p \rangle, \langle q, p+1 \rangle \rangle, 1 \leq j \leq a_p, 1 \leq q \leq a_{p+1}\}$$

with the partial order relation on  $P$  :

$$(x \leq_P y) \iff [(t < v) \vee (t = v \wedge s = u)]$$

for  $x = \langle s, t \rangle, y = \langle u, v \rangle$  being elements of cobweb poset  $P$ . Similary as above one can show that the family of cobweb posets consist of DAGs representable by corresponding  $\dim$  2 posets (i.e. of oDAGs).

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